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On the chaoticity of some tensor product weighted backward shift operators acting on some tensor product Fock-Bargmann spaces

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Abstract

In *Advances in Mathematical Physics* (2011) we showed that the weighted shift $z^p \frac{d^{p+1}}{dz^{p+1}}$ ($p = 0, 1, 2, \dots$) acting on classical Bargmann space \mathbb{B}_p is chaotic operator.

In *Journal of Mathematical physics* (2014), we constructed an chaotic weighted shift $\mathbb{M}^{*p} \mathbb{M}^{p+1}$ ($p = 0, 1, 2, \dots$) on some lattice Fock-Bargmann \mathbb{E}_p^α generated by the orthonormal basis $e_m^{(\alpha,p)}(z) = e_m^\alpha; m = p, p+1, \dots$ where

$$e_m^\alpha(z) = \left(\frac{2\nu}{\pi}\right)^{1/4} e^{\frac{\nu}{2}z^2} e^{-\frac{\pi^2}{\nu}(m+\alpha)^2 + 2i\pi(m+\alpha)z}; m \in \mathbb{N} \text{ with } \nu, \alpha \text{ are real numbers; } \nu > 0,$$

\mathbb{M} is an weighted shift and \mathbb{M}^* is the adjoint of the \mathbb{M} .

In this paper we study the chaoticity of tensor product $\mathbb{M}^{*p} \mathbb{M}^{p+1} \otimes z^p \frac{d^p}{dz^{p+1}}$ ($p = 0, 1, 2, \dots$) acting on $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$. ◇

Keywords: Weighted shift unbounded operators; tensor product operators, chaotic operators; Fock-Bargmann spaces.

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1. Introduction and action of $\mathbb{M}^{*p}\mathbb{M}^{p+1} \otimes z^p \frac{d^{p+1}}{dz^{p+1}}$ on $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$

Let $z = x + iy$; $z \in \mathbb{C}$ and $\nu > 0$, α are fixed real numbers.

We consider the space

$$\mathcal{O}_\alpha(\mathbb{C}) = \{\phi : \mathbb{C} \rightarrow \mathbb{C} \text{ entire}; \psi(z+m) = e^{2i\pi\alpha m} e^{\nu(z+\frac{m}{2})m} \psi(z), \forall z \in \mathbb{C}, \forall m \in \mathbb{N}\}$$

and the Hilbert space

$$\mathbb{E}^\alpha = \{\psi \in \mathcal{O}_\alpha(\mathbb{C}); \int \int_{[0,1] \times \mathbb{R}} |\psi(z)|^2 e^{-\nu|z|^2} dx dy < \infty\} \quad (1.1)$$

with inner product,

$$\langle \psi_1, \psi_2 \rangle_{\mathbb{E}^\alpha} = \int \int_{[0,1] \times \mathbb{R}} \psi_1(z) \overline{\psi_2(z)} e^{-\nu|z|^2} dx dy \quad (1.2)$$

and norm,

$$\|\psi\|_{\mathbb{E}^\alpha} = \sqrt{\int \int_{[0,1] \times \mathbb{R}} |\phi(z)|^2 e^{-\nu|z|^2} dx dy} \quad (1.3)$$

This space is a particular case of (Γ, χ) -theta Fock-Bargmann spaces recently constructed by Ghanmi-Intissar in [9] where it is showed that

$$e_m^\alpha(z) = \left(\frac{2\nu}{\pi}\right)^{1/4} e^{\frac{\nu}{2}z^2} e^{-\frac{\pi^2}{\nu}(m+\alpha)^2 + 2i\pi(m+\alpha)z}; m \in \mathbb{N} \quad (1.4)$$

is orthonormal basis of \mathbb{E}^α . ◇

Remark 1.1 (fundamental)

The explicit construction of orthonormal basis (1.4) of \mathbb{E}^α play a fundamental role to know if an operator acting on \mathbb{E}^α (in which the polynomials are dense), can be represented as a weighted backward shift. ◇

On \mathbb{E}^α , we considered in [14] the weight backward shift operator \mathbb{M} defined by

$\mathbb{M}e_m^\alpha = \gamma_{m-1}e_{m-1}^\alpha, m \in \mathbb{N}; \quad \mathbb{M}e_0^\alpha = 0$ where $\gamma_m = c_\alpha e^{\frac{2\pi}{\nu}m}$ and $c_\alpha = e^{\frac{\pi}{\nu}+2\alpha}$ and we showed the chaoticity of the operator $\mathbb{M}^{*p}\mathbb{M}^{p+1}; p = 0, 1, 2, \dots$ on \mathbb{E}_p^α where

$\mathbb{E}_p^\alpha = \{\phi \in \mathbb{E}^\alpha; \phi(0) = \phi'(0) = \dots \phi^{p-1}(0) = 0\}$, an orthonormal basis of this space is given by

$$e_m^{(\alpha,p)}(z) = e_m^\alpha(z); m = p, p+1, \dots \quad (1.5)$$

- \mathbb{M}^* is adjoint of \mathbb{M}

and

$$\mathbb{M}^{*p} \mathbb{M}^{p+1} e_m^{\alpha,p}(z) = \gamma_{m-1} \left[\prod_{j=1}^p \gamma_{m-1-j} \right]^2 e_{m-1}^\alpha(z) \quad (1.6)$$

e.g the operator $\mathbb{M}^{*p} \mathbb{M}^{p+1}$ verifies the conditions of the following definition

Definition 1.2

A linear unbounded densely defined operator $(\mathbb{T}, D(\mathbb{T}))$ on a Banach space \mathbb{X} is called chaotic or Devaney chaotic if the following conditions are met:

1) \mathbb{T}^n is closed for all positive integers n .

2) there exists an element $\psi \in D(\mathbb{T})^\infty$ whose orbit $Orb(\mathbb{T}, \psi) = \{\psi, \mathbb{T}\psi, \mathbb{T}^2\psi, \dots\}$ is dense in \mathbb{X} where $D(\mathbb{T})^\infty = \cap_{n=0}^\infty D(\mathbb{T}^n)$; such a vector ψ is called a hypercyclic vector for \mathbb{T} , the name hypercyclic was motivated by the concept of a cyclic vector from operator theory. In other words, there is no proper closed \mathbb{T} -invariant subset of \mathbb{X} containing ψ .

3) the set $\{\psi \in \mathbb{X}; \exists m \in \mathbb{N} \text{ such that } \mathbb{T}^m \psi = \psi\}$ of periodic points of operator \mathbb{T} is dense in \mathbb{X} . \diamond

In the sequel of this paper, the orbit and the set of periodic points of operator $\mathbb{M}^{*p} \mathbb{M}^{p+1}$ are denoted by Orb_α and $\mathbb{U}_{per,\alpha}$ respectively (they are dense in \mathbb{E}_p^α). \diamond

Now let \mathbb{B}_p ($p = 0, 1, \dots$) be the classical Bargmann space defined as a subspace of the space $O(\mathbb{C})$ of holomorphic functions on \mathbb{C} such that

$$\mathbb{B}_p = \{\phi \in O(\mathbb{C}); \phi(0) = \phi'(0) = \dots \phi^{p-1}(0) = 0 \text{ and } \langle \phi, \phi \rangle_{\mathbb{B}_p} < \infty\} \quad (1.7)$$

where the pairing $\langle, \rangle_{\mathbb{B}_p}$ is given by

$$\langle \phi_1, \phi_2 \rangle_{\mathbb{B}_p} = \int_{\mathbb{C}} \phi_1(z) \overline{\phi_2(z)} e^{-|z|^2} dx dy \quad (1.8)$$

for all $\phi_1, \phi_2 \in O(\mathbb{C})$ and Lebesgue measure $dx dy$ on \mathbb{C} . \diamond

It is easy to verify that the pairing (1.8) defined on the Bargmann space \mathbb{B}_p ($p = 0, 1, \dots$) is an inner product and the associated norm is

$$\|\phi\|_{\mathbb{B}_p} = \sqrt{\int_{\mathbb{C}} |\phi(z)|^2 e^{-|z|^2} dx dy} \quad (1.9)$$

Now, we can use a theorem of Weierstrass to show that any Cauchy sequence in \mathbb{B}_p has a limit $\phi \in O(\mathbb{C})$ and we check that $\phi \in \mathbb{B}_p$ and indeed is the limit of the Cauchy sequence in the norm $\|\cdot\|_{\mathbb{B}_p}$ of \mathbb{B}_p induced by the inner product. These steps show that the space \mathbb{B}_p is complete and we have

i) The classical Bargmann space \mathbb{B}_p is a Hilbert space.

ii) An orthonormal basis of \mathbb{B}_p is given by

$$e_n^p(z) = \frac{z^n}{\sqrt{n!}}; n = p, p+1, \dots \quad (1.10)$$

On \mathbb{B}_p which is the orthogonal of span $\{e_n^p; n < p\}$ in Bargmann space[3]

$$\mathbb{B}_0 = \{\phi : \mathbb{C} \rightarrow \mathbb{C} \text{ entire} ; \int_{\mathbb{C}} |\phi(z)|^2 e^{-|z|^2} dx dy\} \quad (1.11)$$

with its usual basis:

$$e_n(z) = \frac{z^n}{\sqrt{n!}}; n = 0, 1, \dots \quad (1.12)$$

We considered in [15] the annihilator operator $\frac{d}{dz}$ defined by

$\frac{d}{dz}e_n = \omega_{n-1}e_{n-1}, n \in \mathbb{N}; e_{-1} = 0$ where $\omega_n = \sqrt{n+1}$ and we showed the chaoticity of the operator $z^p \frac{d^p}{dz^{p+1}}; p = 0, 1, 2, \dots$ where z is adjoint of $\frac{d}{dz}$ and

$$z^p \frac{d^p}{dz^{p+1}} e_n^p(z) = \omega_{n-1} \left[\prod_{j=1}^p \omega_{n-1-j} \right]^2 e_{n-1}^p(z) \quad (1.13)$$

e.g the operator $z^p \frac{d^p}{dz^{p+1}}$ verifies the conditions of the definition 1.2 ◇

In the sequel of this paper, the orbit and the set of periodic points of operator $z^p \frac{d^p}{dz^{p+1}}$ are denoted by Orb_p and $\mathbb{U}_{per,p}$ respectively (they are dense in \mathbb{B}_p).

\mathbb{E}_p^α and \mathbb{B}_p are two Hilbert spaces then the tensor product of \mathbb{E}_p^α and \mathbb{B}_p is a new Hilbert space $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$. (see, e.g., [28, Theorem 3.12(b)]). ◇

The reader is referred to Schatten [27] for the theory of cross-spaces and Kubrisly [20] for a concise introduction to tensor product of bounded operators or to Reed-Simon [23] for tensor products of closed operators on Banach spaces.

Below we list a few remarks concerning properties of tensor products, which we will use in the sequel. ◇

Define the elementary elements of the space $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$ as pairs of $\psi \in \mathbb{E}_p^\alpha$ and $\phi \in \mathbb{B}_p$ and written as $\psi \otimes \phi$ where

$$\psi \otimes \phi: \mathbb{E}_p^\alpha \times \mathbb{B}_p \rightarrow \mathbb{C}$$

$$(f, g) \rightarrow \psi \otimes \phi(u, v) = \langle \psi, u \rangle_{\mathbb{E}_p^\alpha} \langle \phi, v \rangle_{\mathbb{B}_p} \quad (1.14)$$

$\psi \otimes \phi$ is called single tensor product and we observe that the single $0 \otimes 0$ coincides with $\psi \otimes 0$ and $0 \otimes \phi$ and the natural map $(\psi, \phi) \rightarrow \psi \otimes \phi$ is not injective.

For $\lambda \in \mathbb{C}$, one identifies $\lambda(\psi \otimes \phi) = (\lambda\psi) \otimes \phi = \psi \otimes (\lambda\phi)$, and considers formal sums of vectors of these elementary vectors. One takes the inner product of two elementary vectors in $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$ as the product of the corresponding inner products,

$$\langle \psi_1 \otimes \phi_1, \psi_2 \otimes \phi_2 \rangle_{\mathbb{E}_p^\alpha \otimes \mathbb{B}_p} = \langle \psi_1, \psi_2 \rangle_{\mathbb{E}_p^\alpha} \cdot \langle \phi_1, \phi_2 \rangle_{\mathbb{B}_p} \quad (1.15)$$

One extends this definition by linearity to finite sums of mn elementary vectors

$$\Phi = \sum_{i=p}^m \sum_{j=p}^n a_{ij} \psi_i \otimes \phi_j, \text{ where } a_{mn} \in \mathbb{C}$$

$$\text{Let } \Psi = \sum_{k=p}^{m'} \sum_{l=p}^{n'} b_{kl} \psi_i \otimes \phi_j, \text{ where } b_{m'n'} \in \mathbb{C}$$

The inner product of two such vectors Ψ and Φ must be linear in Ψ and conjugate linear in Φ

Thus the inner product must have the form:

$$\langle \Psi, \Phi \rangle_{\mathbb{E}_p^\alpha \otimes \mathbb{B}_p} = \sum_{i=p}^m \sum_{j=p}^n \sum_{k=p}^{m'} \sum_{l=p}^{n'} \bar{c}_{ij} c_{kl} \langle \psi_k, \psi_l \rangle_{\mathbb{E}_p^\alpha} \cdot \langle \phi_m, \phi_n \rangle_{\mathbb{B}_p} \quad (1.16)$$

The condition that this form make $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$ into a pre-Hilbert space is the statement that $0 \leq \langle \Psi, \Psi \rangle_{\mathbb{E}_p^\alpha \otimes \mathbb{B}_p}$, with vanishing only possible if $\Psi = 0$. In other words, the form (1.8) is positive definite on $(\mathbb{E}_p^\alpha \otimes \mathbb{B}_p) \times (\mathbb{E}_p^\alpha \otimes \mathbb{B}_p)$. In this case, the algebraic tensor product $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$ is a pre-Hilbert space that can be completed to a Hilbert space that we call $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$. \diamond

In this work as $e_m^\alpha \in \mathbb{E}_p^\alpha$ is an orthonormal basis of \mathbb{E}_p^α and $e_n^p \in \mathbb{B}_p$ is an orthonormal base of \mathbb{B}_p then $e_m^\alpha \otimes e_n^p$ is an orthonormal basis for $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$.

Now let be tow linear operators T_1 with domain $D(T_1)$ on \mathbb{E}_p^α and T_2 with domain $D(T_2)$ on \mathbb{B}_p respectively, we define the tensor product operator $T_1 \otimes T_2$ of T_1 and T_2 on $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$ by:

$$(T_1 \otimes T_2)(\psi \otimes \phi) = (T_1\psi) \otimes (T_2\phi), \quad (1.17)$$

$$\psi \in D(T_1) \text{ and } \phi \in D(T_2)$$

and extends this definition by linearity to all of $D(T_1) \otimes D(T_1)$. As a consequence by adapting Theorem 7.18 in [8] to unbounded operators we get,

$$\text{i) } (T_1 \otimes T_2)(T'_1 \otimes T'_2) = (T_1 T'_1) \otimes (T_2 T'_2) \quad (1.18)$$

on $D(T_1 T'_1) \otimes D(T_1 T'_1)$

$$\text{ii) } (T_1 \otimes T_2)^* = (T_1^* \otimes T_2^*) \quad (1.19)$$

on $D((T_1 \otimes T_2)^*) \cap D(T_1^*) \otimes D(T_2^*)$

$$\text{iii) } (T_1 \otimes T_2)^*(T_1 \otimes T_2) = (T_1^* T_1) \otimes (T_2^* T_2) \quad (1.20)$$

on $D((T_1 \otimes T_2)^*(T_1 \otimes T_2)) \cap D(T_1^* T_1) \otimes D(T_2^* T_2)$.

The matrix elements of $T_1 \otimes T_2$ in the basis $\{e_m^{\alpha,p} \otimes e_n^p\}$ can be expressed in terms of the matrix elements of T_1 in the basis $\{e_m^{\alpha,p}\}$ and T_2 in the basis $\{e_n^p\}$. \diamond

In this paper we are concerned with the problem of preserving properties of chaoticity by tensor product of weighted shift $\mathbb{M}^{*p}\mathbb{M}^{p+1}$ on \mathbb{E}_p^α with weighted shift $z^p \frac{d^p}{dz^{p+1}}$ on \mathbb{B}_p .

Remark 1.3

We cannot use the machinery developed by Reed-Simon [23] because the operators $\mathbb{T}_1 := \mathbb{M}^{*p}\mathbb{M}^{p+1}$ and $\mathbb{T}_2 := z^p \frac{d^p}{dz^{p+1}}$ have empty resolvent sets on \mathbb{E}_p^α and \mathbb{B}_p because their spectrum $\sigma(\mathbb{T}_1) = \sigma(\mathbb{T}_2) = \mathbb{C}$.

Then we show in next section that the operator $\mathbb{T}_1 \otimes \mathbb{T}_2$ verifies the conditions of definition 1.2 of the chaoticity on $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$. \diamond

2. On the chaoticity of $\mathbb{M}^{*p}\mathbb{M}^{p+1} \otimes z^p \frac{d^{p+1}}{dz^{p+1}}$ on $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$

In this section, following [14] and [15] we recall that the operators $\mathbb{T}_1 := \mathbb{M}^{*p}\mathbb{M}^{p+1}$ and $\mathbb{T}_2 := z^p \frac{d^p}{dz^{p+1}}$ are chaotic on \mathbb{E}_p^α and on \mathbb{B}_p respectively and we are concerned with the problem of preserving of this property by tensor product of \mathbb{T}_1 and \mathbb{T}_2 . \diamond

The study of the phenomenon of hypercyclicity originates in the papers by Birkoff [5] and Maclane [21] that show, respectively, that the operators of translation and differentiation, acting on the space of entire functions are hypercyclic.

The theories of hypercyclic operators and chaotic operators have been intensively developed for bounded linear operator, we refer to [2, 7, 10, 11, 12, 13] and ref-

erences therein. \diamond

Remark 2.1

i) For a bounded operator, Ansari asserts in [1] that powers of a hypercyclic bounded operator are also hypercyclic.

ii) In [25] Salas asserts that Weighted backward shifts constitute an important class of operators which is the favorite testing ground for hypercyclicity and characterizes hypercyclicity of weighted backward shift \mathbb{T}_ω acting on

$$l_p = \{(x_n)_{n=0}^\infty \in \mathbb{C}; \sum_{n=0}^\infty |x_n|^p < \infty\} \quad (1 \leq p < +\infty \text{ or } p = 0) \quad (2.1)$$

where \mathbb{T}_ω is defined by:

$$\mathbb{T}_\omega(x_0, x_1, x_2, \dots) = (\omega_1 x_1, \omega_2 x_2, \omega_3 x_3, \dots) \quad (2.2)$$

where $(\omega_1, \omega_2, \omega_3, \dots)$ is a sequence of numbers.

then

a) \mathbb{T}_ω is well defined and continuous if and only if $(\omega_i)_{i=1}^\infty \in l_\infty$

b) \mathbb{T}_ω is hypercyclic on l_p if and only if $\sup_{n \in \mathbb{N}} \prod_{i=1}^\infty \omega_i = \infty$. \diamond

In [22], Martinez-Gimenèz and Peris assert, on universality and chaos for tensor products of bounded weighted backward shift operators, the following proposition

Proposition 2.2 [22]

Let $1 \leq p, q \leq \infty$ and let $\mathbb{T}_\omega : l_p \rightarrow l_p$; $\mathbb{T}_\varpi : l_q \rightarrow l_q$ be two bounded weighted backward shifts. Then $\mathbb{T}_\omega \otimes \mathbb{T}_\varpi : l_p \otimes l_q \rightarrow l_p \otimes l_q$ is hypercyclic on $l_p \otimes l_q$ if and only if $\sup_{n \in \mathbb{N}} \prod_{i=1}^\infty |\omega_i \varpi_i| = \infty$. \diamond

Remark 2.3

The tensor product of two hypercyclic operators is not necessarily hypercyclic. \diamond

It is sufficient to take the pair of weights shifts defined by

$$\mathbb{T}_\omega(x_0, x_1, x_2, \dots) = (2x_1, \frac{1}{2}x_2, \frac{1}{2}x_3, 2x_4, 2x_5, \dots) \quad (2.3)$$

and

$$\mathbb{T}_{\varpi}(x_0, x_1, x_2, \dots) = \left(\frac{1}{2}x_1, 2x_2, 2x_3, \frac{1}{2}x_4, \frac{1}{2}x_5, \dots\right) \quad (2.4)$$

so by using the characterization of Salas [25] we deduce that \mathbb{T}_{ω} and \mathbb{T}_{ϖ} are hypercyclic. Clearly $\sup_{n \in \mathbb{N}} \prod_{i=1}^{\infty} |\omega_i \varpi_i| = 1$ for all $n \in \mathbb{N}$ and by the above proposition, we obtain that $\mathbb{T}_{\omega} \otimes \mathbb{T}_{\varpi}$ is not hypercyclic. \diamond

Remark 2.4

i) For an unbounded operator, Salas exhibit in [24] an unbounded hypercyclic operator whose square is not hypercyclic.

ii) Let \mathbb{B} the classical Bargmann space with orthonormal basis

$$\{e_n = \frac{z^n}{\sqrt{n!}}; n = 0, 1, \dots\}$$

Define the lowering and raising operators \mathbb{A} and \mathbb{A}^* as

$$\mathbb{A}e_n = \sqrt{n}e_{n-1}, \quad \mathbb{A}e_0 = 0 \quad (\text{lowering operator}) \quad (2.5)$$

$$\mathbb{A}^*e_n = \sqrt{n+1}e_{n+1}, \quad (\text{raising operator}) \quad (2.6)$$

a) It is well known that the annihilator operator \mathbb{A} acting on classical Bargmann space is chaotic see [15] but $\mathbb{A}^*\mathbb{A}$ and $\mathbb{A}^* + \mathbb{A}$ are not chaotic where \mathbb{A}^* is the creator operator.

b) So it is well known that the operators $\mathbb{A}^*\mathbb{A}^2$ and $\mathbb{A}^*\mathbb{A}^2 + \mathbb{A}^{*2}\mathbb{A}$ acting on classical Bargmann space B_p ; $p = 0$ are chaotic see [6].

Then

there exist some operators \mathbb{T}_1 and \mathbb{T}_2 acting on Bargmann space such that \mathbb{T}_1 is chaotic and \mathbb{T}_2 is not chaotic with $\mathbb{T}_1 + \mathbb{T}_2$ is chaotic, it suffices to take $\mathbb{T}_1 = \mathbb{A}^*\mathbb{A}^2$ and $\mathbb{T}_2 = \mathbb{A}^{*2}\mathbb{A}$. An complete scattering analysis on $\mathbb{A}^*\mathbb{A}^2 + \mathbb{A}^{*2}\mathbb{A}$ acting on Bargmann space is given in [17].

or

\mathbb{T}_1 and \mathbb{T}_2 acting on Bargmann space such that \mathbb{T}_1 is chaotic and \mathbb{T}_2 is not chaotic with $\mathbb{T}_1 + \mathbb{T}_2$ is not chaotic, it suffices to take $\mathbb{T}_1 = i(\mathbb{A}^{*2}\mathbb{A} + \mathbb{A}^*\mathbb{A}^2)$ with $i^2 = -1$ and $\mathbb{T}_2 = \mathbb{A}^*\mathbb{A}$. An complete spectral analysis is given in [18] and [19].

c) The polynomial operators $P(\mathbb{A}^*, \mathbb{A})$ acting on classical Bargmann space are an excellent laboratory for the study the phenomenons of hypercyclicity or of chaoticity.

d) Generally it is observed that many properties of concept of tensor universality criterion for a sequence of bounded operators are not applicable to a sequence of unbounded operators, in particular to our operators.

The above results show that one must be careful in the formal manipulation of operators with restricted domains. For such operators it is often more convenient to work with vectors rather than with operators themselves. \diamond

We will denote by $D(\mathbb{T}_1) \otimes D(\mathbb{T}_2)$ the set of linear linear combinations of vectors of the $\psi \otimes \phi$ where $\psi \in D(\mathbb{T}_1)$ and $\phi \in D(\mathbb{T}_2)$.

As $D(\mathbb{T}_1)$ and $D(\mathbb{T}_2)$ are dense in \mathbb{E}_p^α and in \mathbb{B}_p respectively then

$D(\mathbb{T}_1) \otimes D(\mathbb{T}_2)$ is dense in $\mathbb{E}_p^\alpha \times \mathbb{B}_p$.

We define $\mathbb{T}_1 \otimes \mathbb{T}_2$ by $(\mathbb{T}_1 \otimes \mathbb{T}_2)(\psi \otimes \phi) = \mathbb{T}_1\psi \otimes \mathbb{T}_2\phi$ (2.7)

and extend by linearity. \diamond

Lemma 2.5

i) The operator $\mathbb{T} := \mathbb{T}_1 \otimes \mathbb{T}_2 = \mathbb{M}^{*p} \mathbb{M}^{p+1} \otimes z^p \frac{d^{p+1}}{dz^{p+1}}$ is closable.

ii) For each positive integer k , the operator \mathbb{T}^k is a closed. \diamond

Proof

i) Let $\Phi \in D(\mathbb{T}_1) \otimes D(\mathbb{T}_2)$ and Ψ is any vector in $D(\mathbb{T}_1^*) \otimes D(\mathbb{T}_2^*)$, then

$$\langle \mathbb{T}_1 \otimes \mathbb{T}_2 \Phi, \Psi \rangle = \langle \Phi, \mathbb{T}_1^* \otimes \mathbb{T}_2^* \Psi \rangle$$

so

$$D(\mathbb{T}_1^*) \otimes D(\mathbb{T}_2^*) \subset D((\mathbb{T}_1 \otimes \mathbb{T}_2)^*).$$

As \mathbb{T}_1 and \mathbb{T}_2 are closable, $D(\mathbb{T}_1^*)$ and $D(\mathbb{T}_2^*)$ are dense.

Therefore, in this case $(\mathbb{T}_1 \otimes \mathbb{T}_2)^*$ is densely defined which proves that $\mathbb{T}_1 \otimes \mathbb{T}_2$ is closable.

ii) As $\mathbb{T}^k := [\mathbb{M}^{*p} \mathbb{M}^{p+1} \otimes z^p \frac{d^{p+1}}{dz^{p+1}}]^k$ is a closed if and only if its graph $\mathfrak{G}(\mathbb{T}^k)$ is a closed linear manifold of $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p \times \mathbb{E}_p^\alpha \otimes \mathbb{B}_p$.

Let $(f_n, \mathbb{T}^k f_n)$ be an sequence witch converges to some (f, g) in $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p \times \mathbb{E}_p^\alpha \otimes \mathbb{B}_p$.

We want to show that $f \in D(\mathbb{T}^k)$ and $g = \mathbb{T}^k f$. To see this, it suffices to take f_n of the form $f_n = \psi_n \otimes \phi_n$ where

$$\begin{aligned} \psi_n \otimes \phi_n: \mathbb{E}_p^\alpha \times \mathbb{B}_p &\rightarrow \mathbb{C} \\ (u, v) &\rightarrow \psi_n \otimes \phi_n(u, v) = \langle \psi_n, u \rangle + \langle \phi_n, v \rangle \end{aligned} \quad (2.8)$$

Then

$\psi_n \otimes \phi_n$ converges to some $\psi \otimes \phi$ in $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$.

In particular $\psi_n(z)$ converges to $\psi(z)$ in \mathbb{C} and $\phi_n(z')$ converges to $\phi(z')$ in \mathbb{C} .

As \mathbb{T}_1^m and \mathbb{T}_2^k are closed, then we deduce that

$$\langle \mathbb{T}_1^k \psi_n, u \rangle \rightarrow \langle \mathbb{T}_1^k \psi, u \rangle \quad \forall \quad u \in \mathbb{E}_p^\alpha \quad (2.9)$$

and

$$\langle \mathbb{T}_2^k \psi_n, v \rangle \rightarrow \langle \mathbb{T}_2^k \psi, v \rangle \quad \forall \quad v \in \mathbb{B}_p \quad (2.10)$$

and

$$\langle \mathbb{T}_1^k \psi_n, u \rangle \langle \mathbb{T}_2^k \psi_n, v \rangle \rightarrow \langle \mathbb{T}_1^k \psi, u \rangle \langle \mathbb{T}_2^k \psi, v \rangle \quad \forall u \in \mathbb{E}_p^\alpha, \forall v \in \mathbb{B}_p \quad (2.11)$$

Now as $\mathbb{T}^k = \mathbb{T}_1^k \otimes \mathbb{T}_2^k$ we deduce from (2.11) that

$\mathbb{T}^k(\psi_n \otimes \phi_n)$ converges to $\mathbb{T}^k(\psi \otimes \phi)$ then $\psi \otimes \phi \in D(\mathbb{T}^k)$ and $g = \mathbb{T}^k(\psi \otimes \phi)$. \diamond

Remark 2.6

a) In the proof of i), we observe that if two unbounded operators are closable then their tensor product is closable also. e.g the property to be closable is preserved by tensor product.

b) We can exhibit a closed operator whose square is not. For example, the operator acting on $L_2[0, 1] \times L_2[0, 1]$ defined by

$$\mathbb{T}(u, v)(x) = (v'(x), f(x)v(0)) \quad \text{with domain } D(\mathbb{T}) = L_2[0, 1] \times H_1[0, 1] \quad (2.12)$$

where $v'(x)$ is the derivative of $v(x)$ and f is a function in $H_1[0, 1]$ with $f(0) = 1$, $H_1[0, 1]$ is the classical Sobolev space.

Then \mathbb{T} , is a closed operator and $D(\mathbb{T}^2) = D(\mathbb{T})$, where $D(\mathbb{T}^2)$ is the domain of \mathbb{T}^2 but the operator \mathbb{T}^2 is not closed and has not closed extension. This operator can, for example, justify the first assumption of the Definition 1.2 for the unbounded linear operators. \diamond

Sufficient conditions for the hypercyclicity of an unbounded operator are given in the following Bès-Chan-Seubert theorem:

Theorem 2.7 (Bès-Chan-Seubert [4])

Let \mathbb{X} be a separable infinite dimensional Banach, and let \mathbb{T} be a densely defined linear operator on \mathbb{X} . Then, \mathbb{T} is hypercyclic if

i) \mathbb{T}^k is a closed operator for all positive integers k

ii) there exists a dense subset \mathbb{F} of the domain $D(\mathbb{T})$ of \mathbb{T} and a (possibly non-linear and discontinuous) mapping $\mathbb{S} : \mathbb{F} \rightarrow \mathbb{F}$ so that $\mathbb{T}\mathbb{S}$ is the identity on \mathbb{F} and $\mathbb{T}^k, \mathbb{S}^k \rightarrow 0$ pointwise on \mathbb{F} as $k \rightarrow +\infty$. \diamond

In [14] and [15] we showed that $\mathbb{T}_1 := \mathbb{M}^{*p}\mathbb{M}^{p+1}$ and $\mathbb{T}_2 := z^p \frac{d^p}{dz^{p+1}}$ are chaotic on \mathbb{E}_p^α and on \mathbb{B}_p in particular they are hypercyclic.

We verify now that the operator $\mathbb{T} = \mathbb{T}_1 \otimes \mathbb{T}_2 = \mathbb{M}^{*p}\mathbb{M}^{p+1} \otimes z^p \frac{d^{p+1}}{dz^{p+1}}$ on $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$ satisfies the hypercyclicity criterion, as quoted above. \diamond

Lemma 2.8

Let $\mathbb{T}_1 = \mathbb{M}^{*p}\mathbb{M}^{p+1}$ with domain $D(\mathbb{T}_1) = \{\psi \in \mathbb{E}_p^\alpha; \mathbb{T}_1\psi \in \mathbb{E}_p^\alpha\}$ where

$$\mathbb{T}_1 e_m^{\alpha,p} = \gamma_m^{\alpha,p} e_{m-1}^{\alpha,p} \text{ with } \gamma_m^{\alpha,p} = \gamma_{m-1} [\prod_{j=1}^p \gamma_{m-1-j}]^2; \gamma_m = c_\alpha e^{\frac{2\pi}{\nu}m} \text{ and}$$

$$c_\alpha = e^{\frac{\pi}{\nu}+2\alpha} \text{ for } m \geq p \geq 0$$

and

$$\mathbb{T}_2 = z^p \frac{d^{p+1}}{dz^{p+1}} \text{ with domain } D(\mathbb{T}_2) = \{\phi \in \mathbb{B}_p; \mathbb{T}_2\phi \in \mathbb{B}_p\} \text{ where}$$

$$\mathbb{T}_2 e_n^p = \omega_n p e_{n-1}^p \text{ with } \omega_n^p = \sqrt{n+1} \frac{n!}{(n-p)!} \text{ for } n \geq p \geq 0$$

Then

$$\mathbb{T} = \mathbb{T}_1 \otimes \mathbb{T}_2 \text{ with domain } D(\mathbb{T}) = D(\mathbb{T}_1) \otimes D(\mathbb{T}_2) \text{ is hypercyclic.} \quad \diamond$$

Proof

Let $\mathbb{F}_\alpha = \{\psi_k = \sum_{m=p}^k a_m e_m^{\alpha,p}\}$ and $\mathbb{F} = \{\psi_k = \sum_{n=p}^k b_n e_n^p\}$ these spaces are dense in \mathbb{E}_p^α and \mathbb{B}_p respectively.

Let $\mathbb{S}_{1,p} : \mathbb{F}_\alpha \rightarrow \mathbb{F}_\alpha$ defined by $\mathbb{S}_{1,p} e_{m,p}^\alpha = \frac{1}{\gamma_m^{\alpha,p}} e_{m+1,p}^\alpha; m \geq p \geq 0$

and

$\mathbb{S}_{2,p} : \mathbb{F} \rightarrow \mathbb{F}$ defined by $\mathbb{S}_{1,p} e_n^\alpha = \frac{1}{\omega_n^p} e_{n+1}^p; n \geq p \geq 0$

then

$$\mathbb{T}_1 \mathbb{S}_{1,p} = \mathbb{I}_{\mathbb{E}_p^\alpha} \text{ and } \mathbb{T}_2 \mathbb{S}_{2,p} = \mathbb{I}_{\mathbb{B}_p} \quad (2.13)$$

a) As $\mathbb{T}_1^k e_{m,p}^\alpha = 0$ for all $k > m \geq p$ and $\mathbb{T}_2^k e_n^p = 0$ for all $k > n \geq p$ we deduce that any element of \mathbb{F}_α can be annihilated by a finite power k_m of \mathbb{T}_1 and any element of \mathbb{F} can be annihilated by a finite power k_n of \mathbb{T}_2

$$\begin{aligned} \text{b) Since as } [\prod_{j=m}^{k_m+m} \gamma_j^{\alpha,p}]^{-1} \text{ and since as } [\prod_{j=n}^{k_n+n} \omega_j^p]^{-1} \text{ we get} \\ \left\{ \begin{array}{l} \mathbb{S}_{1,p} e_{m,p}^\alpha = [\prod_{j=m}^{k_m+m} \gamma_j^{\alpha,p}]^{-1} e_{k+m,p}^\alpha \rightarrow 0 \text{ in } \mathbb{E}_p^\alpha \\ \mathbb{S}_{2,p} e_n^p = [\prod_{j=n}^{k_n+n} \omega_j^p]^{-1} e_{k+n}^p \rightarrow 0 \text{ in } \mathbb{B}_p \end{array} \right. \quad (2.14) \end{aligned}$$

Now let $\mathbb{S} = \mathbb{S}_{1,p} \otimes \mathbb{S}_{2,p}$ and $\mathbb{G} = \mathbb{F}_\alpha \otimes \mathbb{F}$ which is dense in $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$ then

from (2.13) we deduce that

$$\mathbb{T}\mathbb{S} = \mathbb{I}_{\mathbb{F}_\alpha \otimes \mathbb{F}} \quad (2.15) \cup$$

As the single tensor $0 \otimes 0$ coincides with $e_m^\alpha \otimes 0$ and $0 \otimes e_n^p$ then for all $k > \text{Min}(m, n) \geq p$ in particular for $k > \text{Max}(m, n) \geq p$ we have $\mathbb{T}^k e_{m,p}^\alpha \otimes e_n^p = 0$.

then from (2.14), we deduce that any element of \mathbb{G} can be annihilated by a finite power $k_{m,n} = \text{Max}(k_m, k_n)$ of \mathbb{T} and

$$\mathbb{S}^{k_{m,n}} e_{m,p}^\alpha \otimes e_n^p = [\prod_{j=m}^{k_{m,n}+m} \gamma_j^{\alpha,p}]^{-1} [\prod_{j=n}^{k_{m,n}+n} \omega_j^p]^{-1} e_{k+m,p}^\alpha \otimes e_n^p \rightarrow 0 \text{ in } \mathbb{E}_p^\alpha \otimes \mathbb{B}_p \quad (2.16)$$

Now, the hypercyclicity of \mathbb{T} follows from the theorem of Bès et al. recalled above.

Lemma 2.9

Let $\mathbb{T} = \mathbb{T}_1 \otimes \mathbb{T}_2$ with domain $D(\mathbb{T}) = D(\mathbb{T}_1) \otimes D(\mathbb{T}_2)$ where

$\mathbb{T}_1 = \mathbb{M}^{*p} \mathbb{M}^{p+1}$ with domain $D(\mathbb{T}_1) = \{\psi \in \mathbb{E}_p^\alpha; \mathbb{T}_1 \psi \in \mathbb{E}_p^\alpha\}$

$\mathbb{T}_2 = z^p \frac{d^{p+1}}{dz^{p+1}}$ with domain $D(\mathbb{T}_2) = \{\phi \in \mathbb{B}_p; \mathbb{T}_2 \phi \in \mathbb{B}_p\}$

Then

There exist $k > 0$ and $g \in D(\mathbb{T}^k)$ such that $\mathbb{T}^k g = g$. ◇

Proof

Let $(\lambda, \mu) \in \mathbb{C}^2$ and

$$g_{\lambda, \mu} = e_{p,p}^\alpha \otimes e_p^p + \sum_{m=p+1}^{\infty} \sum_{n=p+1}^{\infty} \frac{\lambda^{m-p} \mu^{n-p}}{(\gamma_p^{\alpha,p} \dots \gamma_{m-1}^{\alpha,p})(\omega_p^p \dots \omega_{n-1}^p)} e_{m,p}^\alpha \otimes e_n \quad (2.17)$$

Then $g_{\lambda, \mu} \in D(\mathbb{T})$ and it is an eigenvector of \mathbb{T} associated to eigenvalue $\lambda\mu$

In fact

Let $r > 0$ and $|\lambda\mu| < r$ then $|\lambda| < r$ and $|\mu| < r$

Now, as

$$\lim \prod_{j=p}^{m-1} \gamma_j^{\alpha,p} = +\infty, \quad m \rightarrow +\infty \quad (2.18)$$

and

$$\lim \prod_{j=p}^{n-1} \omega_j^p = +\infty, \quad n \rightarrow +\infty \quad (2.19)$$

then there exist $m_0, n_0 \in \mathbb{N}$, $q < 1$ and $q' < 1$ such that

$$\frac{r}{(\gamma_p^{\alpha,p} \dots \gamma_{m-1}^{\alpha,p})^{\frac{1}{m}}} \leq q \text{ for } m \geq m_0 \quad (2.20)$$

and

$$\frac{r}{(\omega_p^p \dots \omega_{n-1}^p)^{\frac{1}{n}}} \leq q' \text{ for } n \geq n_0 \quad (2.21)$$

As $|\lambda| < r$ and $|\mu| < r$, we deduce that

$$\frac{|\lambda|^{m-p}}{(\gamma_p^{\alpha,p} \dots \gamma_{m-1}^{\alpha,p})^2} \leq q^{2m} \text{ and } \frac{|\mu|^{n-p}}{(\omega_p^p \dots \omega_{n-1}^p)^2} \leq q'^{2n} \text{ for } m \geq m_0, n \geq n_0 \text{ respectively.}$$

As $\{e_{m,p}^\alpha \otimes e_n^p\}$ is orthonormal basis and

$$\sum_{m=p+1}^{\infty} \sum_{n=p+1}^{\infty} \frac{|\lambda|^{m-p} |\mu|^{n-p}}{(\gamma_p^{\alpha,p} \dots \gamma_{m-1}^{\alpha,p})^2 (\omega_p^p \dots \omega_{n-1}^p)^2} < \frac{(qq')^{p+1}}{(1-q^2)(1-q'^2)}$$

then $g_{\lambda,\mu} \in E_p^\alpha \otimes B_p$.

Now as

$$\begin{aligned} \langle g_{\lambda,\mu}, e_{p,p}^\alpha \otimes e_p^p \rangle_{E_p^\alpha \otimes B_p} &= 1 \\ \langle g_{\lambda,\mu}, e_{k+1,p}^\alpha \otimes e_{k+1}^p \rangle_{E_p^\alpha \otimes B_p} &= \frac{\lambda^{m-p} \mu^{n-p}}{(\gamma_p^{\alpha,p} \dots \gamma_{m-1}^{\alpha,p})^2 (\omega_p^p \dots \omega_{n-1}^p)^2} \end{aligned} \quad (2.22)$$

we get

$$|\langle g_{\lambda,\mu}, e_{k+1,p}^\alpha \otimes e_{k+1}^p \rangle_{E_p^\alpha \otimes B_p}|^2 = \frac{\lambda^{2(m-p)} \mu^{2(n-p)}}{(\gamma_p^{\alpha,p} \dots \gamma_{m-1}^{\alpha,p})^2 (\omega_p^p \dots \omega_{n-1}^p)^2} \quad (2.23)$$

and

$$|\langle g_{\lambda,\mu}, e_{k+1,p}^\alpha \otimes e_{k+1}^p \rangle_{E_p^\alpha \otimes B_p}|^2 (\gamma_k^{\alpha,p})^2 (\omega_k^p)^2 = \frac{|\lambda|^{2(k-p)} |\mu|^{2(n-p)}}{(\gamma_p^{\alpha,p} \dots \gamma_{k-1}^{\alpha,p})^2 (\omega_p^p \dots \omega_{k-1}^p)^2} \quad (2.24)$$

and

$$|\langle g_{\lambda,\mu}, e_{k+1,p}^\alpha \otimes e_{k+1}^p \rangle_{E_p^\alpha \otimes B_p}|^2 (\gamma_k^{\alpha,p})^2 (\omega_k^p)^2 \leq (qq')^{2k} |\lambda \mu|^2 \quad (2.25)$$

Then we deduce that $g_{\lambda,\mu} \in D(\mathbb{T})$

Now as $\mathbb{T} = \mathbb{T}_1 \otimes \mathbb{T}_2$ then

$$\mathbb{T} g_{\lambda,\mu} = \mathbb{T}_1 \otimes \mathbb{T}_2 g_{\lambda,\mu} = \lambda \mu g_{\lambda,\mu} \quad (2.26)$$

Therefore $g_{\lambda,\mu}$ is the eigenvector of \mathbb{T} corresponding to the eigenvalue $\lambda \mu$ and it is a periodic point of \mathbb{T} where $\lambda \mu$ is root of unity. \diamond

Lemma 2.9

The set of periodic points of \mathbb{T} is dense in $E_p^\alpha \otimes B_p$.

Proof

Let $\mathbb{U}_{per,\alpha} \subset \mathbb{E}_p^\alpha$ and $\mathbb{U}_{per,p} \subset \mathbb{B}_p$ be the sets of periodic points for \mathbb{T}_1 and \mathbb{T}_2 respectively.

Let $\mathbb{U}_{per} = \mathbb{U}_{per,\alpha} \otimes \mathbb{U}_{per,p}$ which is a subset of set of periodic points of \mathbb{T} . As \mathbb{T}_1 and \mathbb{T}_2 are chaotic then $\mathbb{U}_{per,\alpha}$ and $\mathbb{U}_{per,p}$ are dense in \mathbb{E}_p^α and \mathbb{B}_p respectively hence \mathbb{U}_{per} is a dense subspace of $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$. In particular, The set of periodic points of \mathbb{T} is dense in $\mathbb{E}_p^\alpha \otimes \mathbb{B}_p$.

Now, the chaoticity of \mathbb{T} follows from the above lemmas.

We would like to finish this work with the following remark

Remark 2.10

$$\begin{aligned} \text{Let } z &= (z_1, \dots, z_j, \dots, z_n) \in \mathbb{C}^n; z_j = x_j + iy_j \in \mathbb{C}, 1 \leq j \leq n \\ \mathbb{B} &= \{ \phi : \mathbb{C}^n \rightarrow \mathbb{C} \text{ entire}; \int_{\mathbb{C}^n} |\phi(z)|^2 e^{-|z|^2} \prod_{j=1}^n dx_j \prod_{j=1}^n dy_j < +\infty \} \\ \mathbb{A}_j \phi &= \frac{\partial}{\partial z_j} \phi \text{ with domain } D(\mathbb{A}_j) = \{ \phi \in \mathbb{B}; \mathbb{A}_j \phi \in \mathbb{B} \}, 1 \leq j \leq n \\ \mathbb{A}_j^* \phi &= z_j \phi \text{ with domain } D(\mathbb{A}_j^*) = \{ \phi \in \mathbb{B}; \mathbb{A}_j^* \phi \in \mathbb{B} \}, 1 \leq j \leq n \\ \mathbb{T} &= \sum_{j=1}^n \mathbb{A}_j^* (\mathbb{A}_j + \mathbb{A}_j^*) \mathbb{A}_j \text{ with domain } D(\mathbb{T}) = \{ \phi \in \mathbb{B}; \mathbb{T} \phi \in \mathbb{B} \} \\ \mathbb{B}_j &= \{ \phi_j : \mathbb{C} \rightarrow \mathbb{C} \text{ entire}; \int_{\mathbb{C}} |\phi_j(z_j)|^2 e^{-|z_j|^2} dx_j dy_j < +\infty \} \end{aligned}$$

As $\mathbb{B} = \otimes_{j=1}^n \mathbb{B}_j$ we can write \mathbb{T} under the following form
 $\mathbb{T} = \oplus_{j=1}^n \mathbb{T}_j$ where $\mathbb{T}_j = I_1 \otimes \dots \otimes \mathbb{A}_j \otimes \dots \otimes I_n$

We observe that \mathbb{T} is neither bounded nor self adjoint operator and as the direct sum of two hypercyclic operators is not in general a hypercyclic operator, indeed, Salas [26] showed that there exist hypercyclic operators \mathbb{T}_1 and \mathbb{T}_2 such that the direct sum $\mathbb{T}_1 \oplus \mathbb{T}_2$ is not hypercyclic.

We will give in another paper a comparison of chaoticity of direct sums with chaoticity of tensor products for these operators acting classical Bargmann space or for the operators acting on generalized Fock-Bargmann space like those definite in [16].

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